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On the Free Vibration Analysis of a Plate with an Oblique Inelastic Line Constraint

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The free vibration of a rectangular plate with an oblique inelastic line constraint under various boundary conditions is analytically investigated. A double Fourier sine series is employed for the modal displacement functions and the method of stationary potential energy is applied for the analysis to obtain the general analytical solution. The geometric boundary conditions and interior oblique inelastic line constraint are enforced by means of Stokes' Transformation and the Lagrange multipliers. The normalized frequency parameters and mode shapes are obtained for various boundary conditions and the results are cross-checked with those by MSC/NASTRAN Finite Element package.

Key Words: Free Vibration Analysis, Oblique Inelastic Line Constraint, Double Fourier Sine Series, Stationary Potential Energy, Stokes' Transformation, Lagrange Multipliers, MSC/NASTRAN, Normalized Frequency Parameters, Mode Shapes

Notations

D	: Plate flexural rigidity
a,b	: Plate width and length
h	: Plate thickness
E	: Young's modulus
∇^2	: The Laplacian
$w(x,y)$	: Modal displacement functions of a plate
$f(x,y,t)$	: Transverse displacements of a plate
ν	: Poisson 's ratio
ω	: Circular frequency
$\Omega=a^2\omega\sqrt{h\rho/D}$	: Normalized frequency parameter
ρ	: Mass density
Φ	: Functional
$\delta(\cdots)$	: Variation of the functional
$A_{mn},B_{0n},B_{mn},C_{mn},D_0,D_{0n},D_{m0},D_{mn},E_{m0},E_{mn},E_{0n},F_{mn}$	: Fourier coefficients
c_1,c_2	: End points of an inelastic line constraint
$\Lambda_1,\Lambda_2,\Lambda_{ij}$	: Lagrange multipliers ($j=1,2,\cdots,5$)
$\mu_c,\varepsilon_c,\upsilon_c,\tau_c$	: Aassumed displacements at the corners
$\mu_p,\varepsilon_q,\upsilon_r,\tau_s$	: Fourier coefficients of the assumed displacements at the boundary
θ	: Angle of the inelastic line constraint
U_p	: Strain energy of the plate
K_p	: Kinetic energy of the plate
$U_{\max}$	: Maximum strain energy of the plate
$K_{\max}$	: Maximum kinetic energy of the plate
m,n,l,p,q,r,s	: Fourier numbers
M,N,L,P,Q,R,S	: Number of terms of m, n, l, p, q, r, s
$U-V$	: Transformed coordinate system
$C_{i,j}$	: Elements of the frequency determinant
d	: Length of the inelastic line constraint
F	: Symbol for a free boundary condition
C	: Symbol for a clamped boundary condition

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: Symbol for a simply supported boundary condition

1. Introduction

The free vibration of a rectangular plate has the been subject of numerous studies, many of which has discussed the analysis of the plate which has internal line support. Plate structures of bridge slabs and floor systems which are supported on load bearing walls are used in many engineering applications. The plates involving various complicating factors have been considered, including the case in which the plates sometimes have intermediate inelastic constraints arranged obliquely to the edge of the plate. The complexity of the layout makes it difficult for designers to model the floor systems by the equivalent frame method. A rigorous free vibration analysis of a plate with inelastic line constraint is important to understand the behavior of the plate.

An excellent review of the plate vibration problems with various boundary conditions was made by Leissa (1969). Gorman (1978) used a double Fourier sine series as an analytical free vibration solution for a rectangular plate with an inelastic support along one diagonal. However, the solution was obtained only for a simply supported boundary condition. Takahasi and Chishaki (1978) represented the internal line supports by rows of equidistant point support and calculated the natural frequencies and mode shapes of the plate with simply supported boundary condition. Chung (1981) obtained the natural frequencies and mode shape of the beams with point support under various boundary conditions by using the Stokes' Transformation. Kim and Dickinson (1987), by using orthogonal polynomial functions, studied the flexural vibration of a thin rectangular plate that may be continuous over several supports in one or two directions. These solutions could not deal with boundary conditions which included at least one free boundary condition. In a recent paper by Young and Dickinson (1993), simple polynomials were used with the Rayleigh-Ritz method to obtain the natural frequency parameters of a rectangular plate with an internal line support parallel to the edges. Liew and Xiang (1993) studied the transverse vibration of a thick rectangular Mindlin plate with longitudinal, latitudinal and diagonal line support by the two-dimensional polynomials and Rayleigh-Ritz method.

However, the free vibration analysis of the rectangular plate with an oblique line support, which can be used for the wide range of boundary conditions, including the free boundary condition without assuming specific modal functions has not been published up to authors' best knowledge.

In this paper the analytical solutions for the normalized frequency parameters and mode shapes of a double Fourier sine series type are obtained for the free vibration of a rectangular plate with an inelastic oblique line constraint for various boundary conditions. A unified analytical method is developed, which can be used for a plate with any types of boundary conditions and inelastic oblique line constraint. This method allows freedom in choosing the modal displacement functions and yields analytical solutions for the natural frequencies as well as the mode shapes. It is not necessary to assume new modal displacement functions for each change in boundary condition and intermediate support condition. The use of a double Fourier sine series as modal functions will simplify the free vibration analysis.

In the forced vibration analysis, the spatial distribution of an applied force can be expressed with double Fourier sine series. Also its orthogonal property is a merit for an efficient numerical calculation. To enhance the flexibility of the double Fourier sine series, Lagrange multipliers are utilized to match the interior inelastic oblique line constraint and geometric boundary conditions, and Stokes' Transformation is used to handle the displacements that are not satisfied by the double Fourier sine series.

2. Theory

Consider a rectangular plate with an oblique inelastic line constraint as shown in Fig. 1.

The typical boundary conditions are imposed

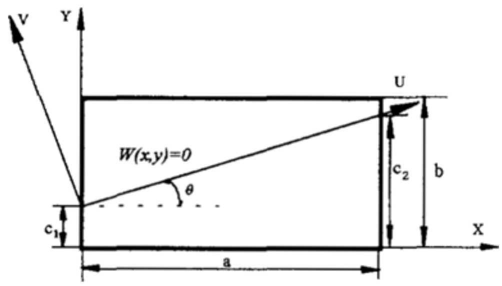

Fig. 1 The geometry and coordinate system of a rectangular plate with an oblique inelastic line constraint.

on $w(x, y)$, $w_{,x}(x, y)$, $w_{,y}(x, y)$, $V_x(x, y)$, $V_y(x, y)$, $M_x(x, y)$, $M_y(x, y)$ at the boundaries, and where $w(x, y)$ are the modal displacement, $w_{,x}(x, y)$ and $w_{,y}(x, y)$ are the derivatives of the modal displacement, $V_x(x, y)$ and $V_y(x, y)$ are the shear forces, and $M_x(x, y)$ and $M_y(x, y)$ are bending moments. The shear forces and bending moments are expressed as follows:

$$V_x = -D[w_{,xxx} + (2-\nu) w_{,xyy}] \quad (1)$$

$$V_y = -D[w_{,yyy} + (2-\nu) w_{,xxy}] \quad (2)$$

$$M_x = w_{,xx} + \nu w_{,yy} \quad (3)$$

$$M_y = w_{,yy} + \nu w_{,xx} \quad (4)$$

where ν is Poisson ratio and D is the plate bending rigidity.

2.1 Method of stationary potential energy

The flexural potential energy of the plate U_p and the corresponding kinetic energy of the plate K_p can be expressed as

$$U_p = \frac{D}{2}\int_0^b \int_0^a \{[\nabla^2 w(x, y)]^2 - 2(1-\nu) [w_{,xx}(x, y) w_{,yy}(x, y) - w_{,xy}^2(x, y)]\} dxdy \quad (5)$$

$$T_p = \frac{\rho h}{2}\int_0^b \int_0^a w_{,t}^2(x, y)\, dxdy \quad (6)$$

where ∇^2 is the Laplacian, ρ is the mass density, a and b are the plate width and length, and h is the plate thickness

The method of stationary potential energy is based on the variational principle $\delta \int_{t1}^{t2} (U_p - T_p)\, dt = 0$. The variation is taken with respect to all the displacements that do not violate the geometric boundary conditions. The shear forces and the bending moments are associated with the natural boundary conditions. The displacements and the slopes of the plate are associated with the geometric boundary conditions.

2.2 Modal displacement functions

If $w(x, y)$ is a modal displacement function that can be expanded in a double Fourier sine series in the region $(0<x<a,\ 0<y<b)$, and if its partial derivatives can be expanded in a cosine -sine series, the coefficients are formed by the usual rule.

$$w(x, y) = \sum_{m=1}^{M}\sum_{n=1}^{N} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \quad (7)$$

$$f(x, y, t) = w(x, y) \sin \omega t \quad (8)$$

where A_{mn} is the Fourier coefficient of $w(x, y)$, m and n are the Fourier number, M and N are the numbers of terms of fourier series (m, n) and $f(x, y, t)$ is the time-varying transverse displacement of the plate.

The assumed modal displacements at the edges and at the corners of a rectangular plate may be represented by

$$w(a, y) = \sum_{p=1}^{P} \mu_p \sin\frac{p\pi y}{b} \quad (0<y<b,\ x=a) \quad (9a)$$

$$w(0, y) = \sum_{q=1}^{Q} \varepsilon_q \sin\frac{q\pi y}{b} \quad (0<y<b,\ x=0) \quad (9b)$$

$$w(x, b) = \sum_{r=1}^{R} \tau_r \sin\frac{r\pi x}{a} \quad (0<x<a,\ y=b) \quad (9c)$$

$$w(x, 0) = \sum_{s=1}^{S} \upsilon_s \sin\frac{s\pi x}{a} \quad (0<x<a,\ y=0) \quad (9d)$$

$$w(0, 0) = \mu_c,\ w(0, b) = \varepsilon_c$$

$$w(a, b) = \tau_c,\ w(a, 0) = \upsilon_c \quad (10)$$

where P, Q, R and are the number of terms in Fourier series (p, q, r, s).

It is not necessary that the double Fourier sine series satisfies any particular boundary conditions, since Lagrange multipliers and Stokes' Transformation can be used to match the appropriate geometric boundary conditions. In the stationary potential energy approach, it is not necessary to enforce the natural boundary conditions but it is necessary to enforce the geometric boundary conditions. The double Fourier sine series in Eq. (7) incidentally satisfies all the geometric boundary conditions for the simply supported plate (S-S-S-S) except the oblique inelastic line constraint and natural boundary conditions. Lagrange multipliers are necessary to enforce the general geometric boundary conditions.

2.3 Stokes' Transformation

By differentiating the double Fourier sine series, the modal displacement function can be expressed as a double Fourier cosine series without the constant term which is not considered to be a complete set of functions. In order to obtain the exact series expressions for the derivatives of a double Fourier sine series, Stokes' Transformation must be utilized. Stokes' Transformation consists of defining each derivative with an independent series and of integrating it by parts and the newly defined series to obtain the relationship between the double Fourier series coefficients:

The successive derivatives of the plate modal displacement functions are as follows:

$$w_{,x}(x,y),\ \sum_{n=1}^{N} B_{0n}\sin\frac{n\pi y}{b}+\sum_{m=1}^{M}\sum_{n=1}^{N} B_{mn}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{11a}$$

$(0\le x\le a,\ 0<y<b)$,

$$w_{,xx}(x,y)=\sum_{m=1}^{M}\sum_{n=1}^{N} C_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{11b}$$

$(0<x<a,\ 0<y<b)$,

$$w_{,xy}(x,y)=D_0+\sum_{m=1}^{M} D_{m0}\cos\frac{m\pi x}{a}+\sum_{n=1}^{N} D_{0n}\cos\frac{n\pi y}{b}+\sum_{m=1}^{M}\sum_{n=1}^{N} D_{mn}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$$

$$(0\le x\le a,\ 0\le y\le b) \tag{11c}$$

$$w_{,y}(x,y)=\sum_{m=1}^{M} E_{m0}\sin\frac{m\pi x}{a}+\sum_{m=1}^{M}\sum_{n=1}^{N} E_{mn}\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b} \tag{11d}$$

$(0<x<a,\ 0\le y\le b)$,

$$w_{,yy}(x,y)=\sum_{m=1}^{M}\sum_{n=1}^{N} F_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \tag{11e}$$

$(0<x<a,\ 0<y<b)$

By using the Stokes' Transformation, the coefficients of derivatives of the modal displacement functions are obtained by integrating by parts of using Eq. (11a) ~ Eq. (11e). The coefficients of the double Fourier series are as follows.

$$B_{0n}=\frac{2}{ab}\int_0^b\left\{\sum_{p=1}^{P}\mu_p\sin\frac{p\pi y}{b}-\sum_{q=1}^{Q}\varepsilon_q\sin\frac{q\pi y}{b}\right\}\sin\frac{n\pi y}{b}dy \tag{12a}$$

$$B_{mn}=\frac{m\pi}{a}A_{mn}+\frac{4}{ab}\int_0^b\left\{\left(\sum_{p=1}^{P}\mu_p\sin\frac{p\pi y}{b}\right)(-1)^m-\sum_{q=1}^{Q}\varepsilon_q\sin\frac{q\pi y}{b}\right\}\sin\frac{n\pi y}{b}dy \tag{12b}$$

$$C_{mn}=-\left(\frac{m\pi}{a}\right)^2 A_{mn} \tag{12c}$$

$$D_0=\tau_c-\varepsilon_c-\upsilon_c+\mu_c \tag{12d}$$

$$D_{0n}=\tau_c(-1)^n-\varepsilon_c-\upsilon_c(-1)^n+\mu_c+\frac{m\pi}{b}\int_0^b\left\{\left(\sum_{p=1}^{P}\mu_p\sin\frac{p\pi y}{b}\right)(-1)^m-\sum_{q=1}^{Q}\varepsilon_q\sin\frac{q\pi y}{b}\right\}dy \tag{12e}$$

$$D_{m0}=\tau_c(-1)^m-\varepsilon_c-\upsilon_c(-1)^m+\mu_c+\frac{m\pi}{a}\int_0^a\left\{\left(\sum_{r=1}^{R}\tau_r\sin\frac{r\pi x}{a}\right)(-1)^n-\sum_{s=1}^{S}\upsilon_s\sin\frac{s\pi x}{a}\right\}dx \tag{12f}$$

$$D_{mn}=\left(\frac{m\pi}{a}\frac{n\pi}{b}\right)A_{mn} \tag{12g}$$

$$E_{m0}=\frac{2}{ab}\int_0^a\left\{\sum_{r=1}^{R}\tau_{r},\sin\frac{r\pi x}{a}-\sum_{s=1}^{S}\upsilon_{s},\sin\frac{s\pi x}{a}\right\}\sin\frac{m\pi x}{a}dx \tag{12h}$$

$$E_{mn}=\frac{n\pi}{b}A_{mn}+\frac{4}{ab}\int_0^a\left\{\left(\sum_{r=1}^{R}\tau_{r},\sin\frac{r\pi x}{a}\right)(-1)^n-\sum_{s=1}^{S}\upsilon_s\sin\frac{s\pi x}{a}\right\}\sin\frac{m\pi x}{a}dx \tag{12i}$$

$$F_{mn}=\left(\frac{n\pi}{b}\right)^2 A_{mn} \tag{12j}$$

2.4 Mathematical formulation

According to the variational principle, the shear force boundary condition needs no explicit consideration since it is natural boundary condition. If the geometric boundary condition is clamped at the edges, the derivatives of the displacements at the edges must be forced to zero.

$$w_{,x}(a,y)=\sum_{m=1}^{M}\sum_{n=1}^{N} B_{mn}(-1)^m\sin\frac{n\pi y}{b}=0 \tag{13a}$$

$$w_{,x}(0,y)=\sum_{m=1}^{M}\sum_{n=1}^{N} B_{mn}\sin\frac{n\pi y}{b}=0 \tag{13b}$$

$$w_{,y}(x,b)=\sum_{m=1}^{M}\sum_{n=1}^{N} E_{mn}(-1)^n\sin\frac{m\pi x}{a}=0 \tag{13c}$$

$$w_{,y}(x,0)=\sum_{m=1}^{M}\sum_{n=1}^{N} E_{mn}\sin\frac{m\pi x}{a}=0 \tag{13d}$$

The geometric boundary conditions that must be forced to zero are the end points of the oblique inelastic constraint line.

$$w(0,\ c_1)=\sum_{q=1}^{Q}\varepsilon_q \sin\frac{q\pi c_1}{b}=0 \tag{14a}$$

$$w(a,\ c_2)=\sum_{p=1}^{P}\mu_p \sin\frac{p\pi c_2}{b}=0 \tag{14b}$$

where c_1 and c_2 are the ends of oblique inelastic line constraint.

It is straightforward to have the coordinate system transformation rule between X-Y and U-V (see Fig. 1). The U-V is the coordinates of the oblique inelastic line constraint in the X-Y coordinate system.

The coordinates of the inelastic line constraint and in U axis are as follows:

$$\tilde{x}=u\cos\theta,\ \tilde{y}=c_1+u\sin\theta \tag{15}$$

where θ is the angle of the oblique inelastic line constraint.

The displacement $w(\tilde{x},\ \tilde{y})$ along the oblique inelastic line constraint that lies in the U-V coordinates is as follows.

$$w(\tilde{x},\ \tilde{y})=\sum_{m=1}^{M}\sum_{n=1}^{N}A_{mn}\sin\frac{m\pi\tilde{x}}{a}\sin\frac{n\pi\tilde{y}}{b}=0 \tag{16a}$$

$$w(\tilde{x},\ \tilde{y})=\sum_{m=1}^{M}\sum_{n=1}^{N}A_{mn}\sin\frac{m\pi(u\cos\theta)}{a}\sin\frac{n\pi(c_1+u\sin\theta)}{b}=0 \tag{16b}$$

$$(0\le u\le d)$$

where d is the length of the oblique inelastic line constraint.

The variational procedure then involves minimization of the following functional Φ to be made stationary:

$$\begin{aligned}\Phi=U_{\max}-T_{\max}&+\int_0^b\left\{\sum_{l=1}^{L}\Lambda_{l1}\cos\frac{lpy}{b}w_{,x}(a,\ y)\right\}dy\\&+\int_0^b\left\{\sum_{l=1}^{L}\Lambda_{l2}\cos\frac{lpy}{b}w_{,x}(0,\ y)\right\}dy\\&+\int_0^a\left\{\sum_{l=1}^{L}\Lambda_{l3}\cos\frac{lpx}{a}w_{,y}(x,\ b)\right\}dx\\&+\int_0^a\left\{\sum_{l=1}^{L}\Lambda_{l4}\cos\frac{lpx}{a}w_{,y}(x,\ 0)\right\}dx\\&+\int_0^d\left\{\sum_{l=1}^{L}\Lambda_{l5}\sin\frac{lpu}{a}w(\tilde{x},\ \tilde{y})\right\}du\\&+\Lambda_1\left\{\sum_{q=1}^{Q}\varepsilon_q\sin\frac{qpc_1}{b}\right\}+\Lambda_2\left\{\sum_{p=1}^{P}\mu_p\sin\frac{ppc_2}{b}\right\}\end{aligned} \tag{17}$$

where l is the Fourier numbers of Lagrange multipliers and L is the number of terms(l) in Lagrange multipliers.

Lagrange multipliers (Λ_1, Λ_2, Λ_{l5}) are related to the force type quantities and required to enforce the inelastic oblique line constraint. Other Lagrange multipliers (Λ_{l1}, Λ_{l2}, Λ_{l3}, Λ_{l4}) are related to the moment type quantities and required to enforce the zero derivatives along the boundary. By substitution of the double Fourier sine series and its appropriate derivatives in Eq. (7) and Eq. (8) and Eq. (11a)~Eq. (11e) into Eq. (17), the frequency determinant may now be easily constructed with the stationary conditions of the functional with respect to the quantities (A_{mn}, μ_p, ε_q, ζ_r, υ_s, μ_c, ε_c, τ_c, υ_c, Λ_1, Λ_2, Λ_{l1}, Λ_{l2}, $\cdots$, Λ_{l5}).

The aforementioned equations above obtained from stationary conditions lead to this homogeneous equation.

$$[C_{i,j}]\{A_{mn},\ \mu_p,\ \varepsilon_q,\ \tau_r,\ \upsilon_s,\ \mu_c,\ \varepsilon_c,\ \tau_c,\ \upsilon_c,\ \Lambda_1,\ \Lambda_2,\ \Lambda_{l1},\ \Lambda_{l2},\ \cdots,\ \Lambda_{l5}\}^T=\{0\} \tag{18}$$

$$(i,\ j=1,\ 2,\ 3,\ \cdots,\ M^2N^2+2M+2N+5K+6)$$

where $C_{i,j}$ are the elements of the frequency determinant.

The natural frequencies of the plate are obtained by assuming various frequency domain parameters, starting from a value near zero, to determine the values that make the determinant vanish.

The above mentioned equations above obtained from stationary conditions lead to the following homogeneous equation.

$$|C_{i,j}|=0 \tag{19}$$

The frequency determinant is symmetric, and is composed of the natural frequency.

3. Examples of Frequency Determinants

In Table 2, each letter indicates the boundary condition at edges starting from the $x=a$, $x=0$, $y=b$, $y=0$ order (F: free, S: simply supported, C: clamped).

3.1 F-C-F-F Rectangular plate with an oblique line constraint

Consider the case of a free-clamped-free-free plate. The boundary conditions and the interior inelastic displacement constraint are

$$w_{,x}(0,\ y)=0,\ w(\tilde{x},\ \tilde{y})=0. \tag{20}$$

The frequency determinant is found from Eq. (18) by retaining the rows and columns associated with A_{mn}, μ_p, τ_r, υ_s, τ_c, υ_c, Λ_2, Λ_{l2}, Λ_{l5}. The resulting homogeneous equations and frequency determinant are

$$[C_{i,j}]\{A_{mn},\ \mu_p,\ \tau_r,\ \upsilon_s,\ \tau_c,\ \upsilon_c,\ \Lambda_2,\ \Lambda_{l2},\ \Lambda_{l5}\}^T=\{0\} \tag{21}$$

$(m=r=s=1,\ 2,\ 3,\ \cdots,\ M\ ;\ n=p=1,\ 2,\ 3,\ \cdots,\ N\ ;\ l=1,\ 2,\ \cdots,\ L)$

$$|C_{i,j}|=0(i,\ j=1,\ 2,\ 3,\ \cdots,\ M^2N^2+N+2M+2L+1). \tag{22}$$

3.2 C-C-F-S Rectangular plate with an oblique line constraint

Consider the case of a clamped-clamped-free-simply supported plate. The boundary conditions and the interior inelastic displacement constraint are

$$w_{,x}(0,\ y)=0,\ w_{,x}(a,\ y)=0,\ w(\tilde{x},\ \tilde{y})=0. \tag{23}$$

The frequency determinant is found from Eq. (15) by retaining the rows and columns associated with A_{mn}, τ_r, Λ_{l1}, Λ_{l2}, Λ_{l5}. The resulting homogeneous equation and frequency determinant are

$$[C_{i,j}]\{A_{mn},\ \tau_r,\ \Lambda_{l1},\ \Lambda_{l2},\ \Lambda_{l5}\}^T=\{0\} \tag{24}$$

$(m=r=1,\ 2,\ 3,\ \cdots,\ M\ ;\ n=1,\ 2,\ 3,\ \cdots,\ N\ ;\ l=1,\ 2,\ \cdots,\ L)$

$$|C_{i,j}|=0(i,\ j=1,\ 2,\ 3,\ \cdots\ M^2N^2+M+3L). \tag{25}$$

3.3 S-S-S-S Rectangular plate with an oblique line constraint

Consider the case of a simply supported at each edge. The interior inelastic displacement constraint is

$$w(\tilde{x},\ \tilde{y})=0. \tag{26}$$

The frequency determinant is found from Eq. (18) by retaining the rows and columns associated with A_{mn}, Λ_{l5}. The resulting homogeneous equation and frequency determinant are

$$[C_{i,j}]\{A_{mn},\ \Lambda_{l5}\}^T=\{0\} \tag{27}$$

$(m=n=1,\ 2,\ 3,\ \cdots,\ M\ ;\ n=1,\ 2,\ 3,\ \cdots,\ N\ ;\ l=1,\ 2,\ \cdots,\ L)$

$$|C_{i,j}|=0(i,\ j=1,\ 2,\ 3,\ \cdots,\ M^2N^2+L). \tag{28}$$

3.4 C-C-C-C Rectangular plate with an oblique line constraint

Consider the case of a clamped at each edge. The boundary conditions and the interior inelastic displacement constraint are

$$w_{,x}(a,\ y)=0,\ w_{,x}(0,\ y)=0,\ w_{,y}(x,\ b)=0,$$
$$w_{,y}(x,\ 0)=0,\ w(\tilde{x},\ \tilde{y})=0. \tag{29}$$

The frequency determinant is found from Eq. (18) by retaining the rows and columns associated with A_{mn}, Λ_{l1}, Λ_{l2}, Λ_{l3}, Λ_{l4}, Λ_{l5}. The resulting homogeneous equation and frequency determinant are

$$[C_{i,j}]\{A_{mn},\ \Lambda_{l1},\ \Lambda_{l2},\ \Lambda_{l3},\ \Lambda_{l4},\ \Lambda_{l5}\}^T=\{0\} \tag{30}$$

$(m=1,\ 2,\ 3,\ \cdots,\ M\ ;\ n=1,\ 2,\ 3,\ \cdots,\ N\ ;\ l=1,\ 2,\ \cdots,\ L)$

$$|C_{i,j}|=0(i,\ j=1,\ 2,\ 3,\ \cdots,\ M^2N^2+5L). \tag{31}$$

4. Results and Discussion

The numerical results on the normalized natural frequency and mode shapes can be obtained. The geometry and material properties of the plate with an oblique inelastic line constraint used in this analysis are shown in Table 1.

4.1 Normalized frequency parameters

Normalized frequency parameters are obtained from the frequency determinant by monitoring the determinant until it vanishes. Table 2 shows the lowest 4 normalized natural frequencies Ω for a rectangular plate with an interior displacement constraint under 21 boundary conditions obtained by the present method and commercial F. E. M. package, MSC/NASTRAN. The normalized frequency parameter is $\Omega=a^2\,\omega\sqrt{h\rho/D}$.

The results obtained by the Double Fourier

Table 1. Data of a rectangular plate with an oblique inelastic line constraint.

Material	Mild steel
Width of a plate (a)	1m
Length of a plate (b)	2m
Left end point of inelastic line constraint (c_1)	0.95m
Right end point of inelastic line constraint (c_2)	1.05m
Slope of the inelastic line constraint ($\tan\theta$)	0.1
Young's modulus (E)	200GPa
Thickness of a plate (h)	0.01m
Poisson's ratio (ν)	0.3
Density of a plate (ρ)	7800Kg/m^3

Table 2. Normalized frequency parameters ($\Omega=a^2\omega\sqrt{h\rho/D}$) of a rectangular plate with an oblique inelastic line constraint by analysis and MSC/NASTRAN.
($c_1=0.95$; $c_2=1.05$; $\tan\theta=0.1$; $M=N=P=Q=R=S=40$; $L=30$).

B. C.	Method	Mode sequence			
		1	2	3	4
F-F-F-F	MSC/NASTRAN	3.45	6.60	8.38	14.69
	Present analysis	3.92	6.92	8.92	15.25
F-S-F-S	MSC/NASTRAN	3.36	5.31	16.87	18.64
	Present analysis	3.69	5.47	18.37	19.34
F-C-F-F	MSC/NASTRAN	5.31	6.88	18.59	23.51
	Present analysis	5.62	7.10	20.52	24.71
F-S-F-S	MSC/NASTRAN	4.40	13.08	18.38	22.42
	Present analysis	4.63	13.71	19.,24	23.39
S-S-F-F	MSC/NASTRAN	11.64	12.58	27.13	32.16
	Present analysis	11.74	12.92	28.43	33.55
S-C-F-F	NASTRAN	16.72	17.42	30.35	35.00
	Present analysis	16.77	67.87	31.78	36.61
F-C-F-S	MSC/NASTRAN	6.17	13.90	22.62	24.98
	Present analysis	7.12	14.41	23.78	26.23
C-C-F-F	MSC/NASTRAN	23.32	23.80	34.68	38.81
	Present analysis	22.89	24.08	36.61	40.95
F-C-F-C	MSC/NASTRAN	6.26	17.87	23.56	25.48
	Present analysis	6.21	18.95	24.77	26.45
S-S-F-S	MSC/NASTRAN	12.06	21.07	29.94	41.24
	Present analysis	12.14	21.31	31.18	41.79
S-S-F-C	MSC/NASTRAN	14.83	25.16	32.31	44.68
	Present analysis	14.23	25.46	32.07	42.55

sine series show the maximum error of about 10% for the third mode number of F-C-F-F boundary condition.

Figure 2 shows the convergence of the first two normalized frequency parameters of F-C-F-F rectangular plates as the function of the number of the Lagrange multipliers(L) employed. The rate of convergence of the present results appears to be reasonably rapid with $M=N=30$ terms. Fig. 3 shows the plots of the normalized fre quency parameters of the first two modes of an S-S-S-S rectangular plate with interior displacement constraint with the angle(θ) from 0° to 63.43°(diagonal). In this case, we assume that the

Table 2. (continued)

B. C.	Method	Mode sequence			
		1	2	3	4
C-C-F-S	MSC/NASTRAN	23.53	29.68	36.94	56.54
	Present analysis	24.18	30.20	38.48	58.52
S-C-F-S	MSC/NASTRAN	17.07	24.55	33.11	51.42
	Present analysis	17.37	24.97	34.54	52.40
S-C-F-C	MSC/NASTRAN	16.77	27.75	33.56	50.30
	Present analysis	17.30	28.91	36.01	52.40
C-C-F-C	MSC/NASTRAN	22.67	32.30	36.98	60.16
	Present analysis	24.18	33.85	39.57	64.74
S-S-S-S	MSC/NASTRAN	19.56	23.33	48.31	48.57
	Present analysis	19.74	23.58	49.15	49.34
S-S-S-C	MSC/NASTRAN	20.60	26.76	49.07	51.23
	Present analysis	20.92	27.53	49.94	52.30
S-C-S-C	MSC/NASTRAN	24.37	29.56	53.63	58.40
	Present analysis	24.87	30.29	54.97	60.00
S-S-C-S	MSC/NASTRAN	23.39	28.56	50.74	53.37
	Present analysis	23.98	29.41	52.11	55.07
C-C-S-C	MSC/NASTRAN	29.33	34.00	56.15	68.52
	Present analysis	30.29	35.23	58.13	71.35
C-C-C-C	MSC/NASTRAN	31.41	35.43	62.10	70.07
	Present analysis	32.76	36.81	65.13	71.45

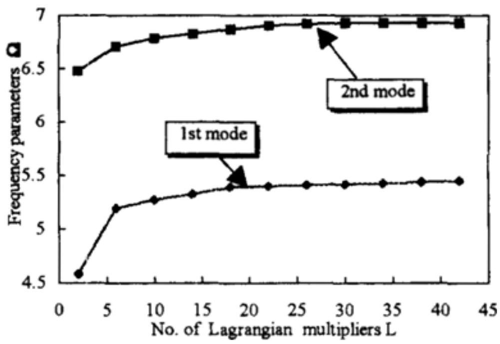

Fig. 2 The normalized frequency parameters of a F-C-F-F rectangular plate with oblique inelastic line constraint vs. no. of Lagrange multipliers.
($C_t = 0.95$; $c_2 = 1.05$; $\tan\theta = 0.1$; $M = N = P = Q = R = S = 40$; $L = 30$)

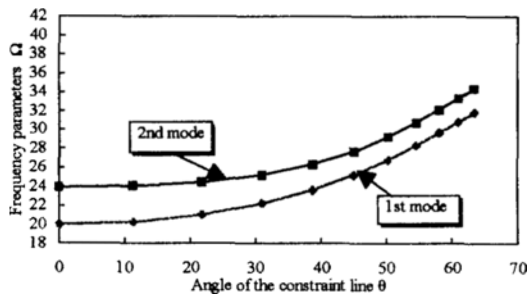

Fig. 3 Normalized frequency parameters of a S-S-S-S rectangular plate with oblique inelastic line constraint vs. the angle (θ) of the oblique inelastic line constraint.
($M = N = 20$; $L = 16$)

center of the inelastic oblique line constraint is always coincident with that of the plate. The frequency parameters increase with the increase of the angle of the inelastic constraint line.

4.2 Mode shapes

Once we obtain the normalized frequency parameters, their corresponding mode shapes can be determined. Figure 4 shows the mode shapes of the first and second modes for the cases of F-C-F-F, C-C-F-S, S-S-S-S and C-C-C-C boundary conditions. The first(fundamental) mode of a vibration will evidently be anti-symmetric since this will involve the least amount of bending

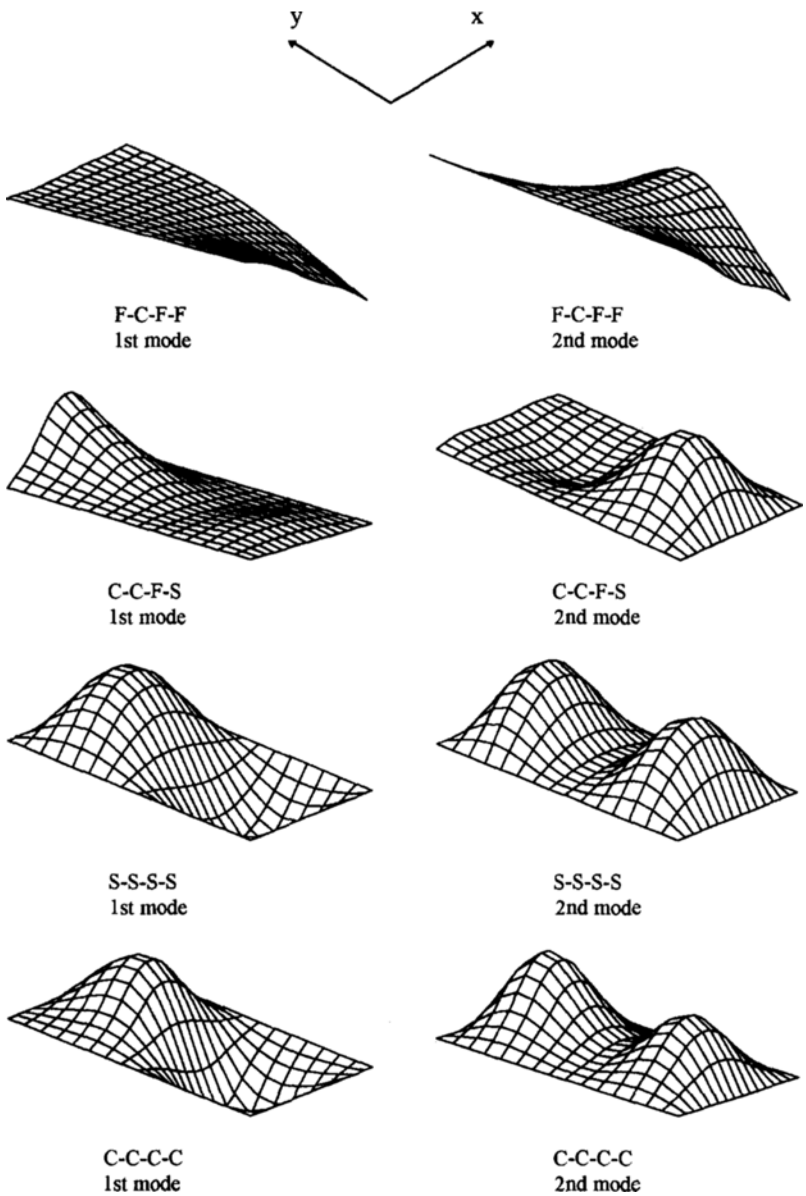

Fig. 4 Mode shapes for the rectangular plate with an oblique inelastic line constraint. ($c_1=0.95$; $c_2=1.05$; $\tan\theta=0.1$; $M=N=P=Q=R=S=40$; $L=30$)

energy for symmetric C-C-C-C boundary condition along the oblique inelastic line constraint.

5. Conclusions

The proposed method for the free vibration analysis of a plate with an oblique inelastic line constraint is found to be effective regardless of the boundary conditions. This method is based on the Rayleigh-Ritz method in which only a double Fourier sine series is used in the modal displacement functions. Some boundary conditions and the interior constraint that cannot be satisfied only with the double Fourier sine series are enforced by means of the Stokes' Transformation and Lagrange multipliers. Further research into the free vibration of the stiffened plate with arbitrarily oblique stiffeners will be carried out in future. The normalized frequency parameters and mode shapes obtained by the presented method are well compared with the numerical results obtained by MSC/NASTRAN.

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Appendix A. Elements of frequency determinants in Eq. (19)

$C_{1,1}=\mathrm{diag}[\Gamma_{mn}]$, $C_{1,2}=[0]$, $C_{1,3}=[0]$, $C_{1,4}=[\Pi_{mn,j l1}]$,

$C_{1,5}=[\Theta_{mn,l2}]$, $C_{1,6}=[\Xi_{mn,13}]$, $C_{1,7}=[\Psi_{mn,14}]$, $C_{1,8}=[Y_{mn,15}]$,

$C_{1,9}=[P1_{mn}]$, $C_{1,10}=[P2_{mn}]$, $C_{1,11}=[P3_{mn}]$, $C_{1,12}=[P4_{m,n}]$,

$C_{1,13}=[Q1_{mn,q}]$, $C_{1,14}=[Q2_{mn,q}]$, $C_{1,15}=[Q3_{mn,r}]$, $C_{1,16}=[Q4_{mn,s}]$,

$$C_{2,2}=\left[\sum_{q=1}^{Q}\varepsilon_q \sin\frac{p\pi c_1}{b}\right],\ C_{2,3}=\left[\sum_{p=1}^{P}\mu_p \sin\frac{q\pi c_2}{b}\right],\ C_{2,i}=[0]\ (i=4,\ 5,\ \cdots,\ 16),$$

$C_{i,j}=[0]\ (i,\ j=3,\ 4,\ \cdots,\ 8)$

$$C_{9,9}=\frac{abD}{8}\left[\frac{16(1-\nu)}{a^2b^2}+\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}+\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}+\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$$C_{9,10}=\frac{abD}{8}\left[\frac{16(1-\nu)}{a^2b^2}-(-1)^n\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}-\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}-(-1)^n\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$$C_{9,11}=\frac{abD}{8}\left[\frac{16(1-\nu)}{a^2b^2}+(-1)^n\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}+(-1)^m\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}+(-1)^{n+m}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$$C_{9,12}=\frac{abD}{8}\left[-\frac{16(1-\nu)}{a^2b^2}-\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}-(-1)^m\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}-(-1)^m\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$C_{9,i}=[0]\,(i=13,\,14,\,\cdots,\,16)$, $C_{10,10}=C_{9,9}$,

$$C_{10,11}=\frac{abD}{8}\left[-\frac{16(1-\nu)}{a^2b^2}-\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}-(-1)^m\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}-(-1)^n\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$$C_{10,12}=\frac{abD}{8}\left[\frac{16(1-\nu)}{a^2b^2}+(-1)^n\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}+(-1)^m\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}+(-1)^{n+m}\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right]$$

$C_{10,i}=[0]\,(i=13,\,14,\,\cdots,\,16)$

$$C_{11,11}=\frac{abD}{8}\left[\frac{16(1-\nu)}{a^2b^2}+\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}+\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}+\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$$C_{11,12}=\frac{abD}{8}\left[-\frac{16(1-\nu)}{a^2b^2}-(-1)^n\sum_{n=1}^{N}\frac{32(1-\nu)}{a^2b^2}-\sum_{m=1}^{M}\frac{32(1-\nu)}{a^2b^2}-(-1)^n\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{64(1-\nu)}{a^2b^2}\right],$$

$C_{11,i}=[0]\,(i=13,\,14,\,\cdots,\,16)$, $C_{12,12}=C_{11,11}$, $C_{12,i}=[0]\,(i=13,\,14,\,\cdots,\,16)$

$C_{13,13}=\mathrm{diag}[R1_{p,p}]$, $C_{13,14}=[R2_{p,q}]$, $C_{13,15}=[R3_{p,r}]$, $C_{13,16}=[R4_{p,s}]$,

$C_{14,14}=\mathrm{diag}[R5_{q,q}]$, $C_{14,15}=[R6_{q,r}]$, $C_{14,16}=[R7_{q,s}]$, $C_{15,15}=[R8_{r,r}]$,

$C_{15,16}=[R9_{r,s}]$, $C_{16,16}=[S10_{s,s}]$, $C_{i,j}=C_{j,i}^T$

where diag $[\cdots]$ is a diagonal matrix

Appendix B. Components of elements in frequency determinants in appendix A

$$\Gamma_{mn}=\frac{abD\pi^4}{4}\left\{\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]-\Omega\right\},$$

$$\Pi_{mn,l1}=\frac{m\pi}{2a}(-1)^m\left\{-\frac{\cos(n\pi-l\pi)}{(n\pi/b-l\pi/b)}-\frac{\cos(n\pi+l\pi)}{(n\pi/b+l\pi/b)}\right\},$$

$$\Theta_{mn,l2}=\frac{m\pi}{2a}\left\{-\frac{\cos(n\pi-l\pi)}{(n\pi/b-l\pi/b)}-\frac{\cos(n\pi+l\pi)}{(n\pi/b+l\pi/b)}\right\}\},$$

$$\Xi_{mn,l3}=\frac{n\pi}{2b}(-1)^n\left\{-\frac{\cos(m\pi-l\pi)}{(m\pi/a-l\pi/a)}-\frac{\cos(m\pi+l\pi)}{(m\pi/a+l\pi/a)}\right\}$$

$$\Psi_{mn,l4}=\frac{n\pi}{2b}\left\{-\frac{\cos(m\pi-l\pi)}{(m\pi/a-l\pi/a)}-\frac{\cos(m\pi+l\pi)}{(m\pi/a+l\pi/a)}\right\},$$

$$Y_{mn,l5}=\frac{1}{4}\left[\begin{array}{l}\cos\dfrac{n\pi c_1}{b}\left\{\begin{array}{l}\dfrac{1-\cos(-m\pi d/a+n\pi d/b+l\pi)}{(-m\pi/a+n\pi/b+l\pi/d)}+\dfrac{1-\cos(m\pi d/a+n\pi d/b-l\pi)}{(m\pi/a+n\pi/b-l\pi/d)}\\ \dfrac{1-\cos(m\pi d/a-n\pi d/b+l\pi}{(m\pi/a-n\pi/b+l\pi/d)}+\dfrac{1-\cos(-m\pi d/a-n\pi d/b-l\pi)}{(-m\pi/a-n\pi/b-l\pi/d)}\end{array}\right\}+\\ \sin\dfrac{n\pi c_1}{b}\left\{\begin{array}{l}\dfrac{\sin(-m\pi d/a+n\pi d/b+l\pi)}{(-m\pi/a+n\pi/b+l\pi/d)}+\dfrac{\sin(m\pi d/a+n\pi d/b-l\pi)}{(m\pi/a+n\pi/b-l\pi/d)}\\ \dfrac{\sin(m\pi d/a+n\pi d/b+l\pi)}{(m\pi/a+n\pi/b+l\pi/d)}+\dfrac{\sin(m\pi d/a-n\pi d/b+l\pi)}{(m\pi/a-n\pi/b+l\pi/d)}\end{array}\right\}\end{array}\right]$$

$$P1_{mn}=\frac{(1-\nu)D}{4ab},\quad P2_{mn}=-\frac{(1-\nu)D}{4ab}(-1)^n,$$

$$P3_{mn}=\frac{(1-\nu)D}{4ab}(-1)^{(m+n)},\quad P4_{mn}=-\frac{(1-\nu)D(-1)^m}{4ab}(-1)^m,$$

$$Q1_{mn,p}=\frac{ab^2D}{16}(-1)^m\left[\frac{8m^3\pi^3}{a^4b}+\frac{(16-8\nu)mn^2\pi^3}{a^2b^3}\right]\delta_{np},$$

$$Q2_{mn,q}=-\frac{ab^2D}{16}\left[\frac{8m^3\pi^3}{a^4b}+\frac{(16-8\nu)mn^2\pi^3}{a^2b^3}\right]\delta_{nq},$$

$$Q3_{mn,r}=\frac{ab^2D}{16}(-1)^n\left[\frac{8n^3\pi^3}{a^4b}+\frac{(16-8\nu)nm^2\pi^3}{b^2a^3}\right]\delta_{mr},$$

$$Q4_{mn,s}=-\frac{ab^2D}{16}\left[\frac{8n^3\pi^3}{b^4a}+\frac{(16-8\nu)nm^2\pi^3}{b^2a^3}\right]\delta_{ms},$$

$$R1_{p,p}=\sum_{n=1}^{N}\frac{(1-\nu)Dn^2\pi^2}{a}\delta_{np}+\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dm^2\pi^2}{a^3}\delta_{np}+\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{2(1-\nu)Dn^2\pi^2}{ab}\delta_{np},$$

$$R2_{p,q}=-\sum_{n=1}^{N}\frac{(1-\nu)Dn^2\pi^2}{a}\delta_{np}\delta_{nq}-\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dm^2\pi^2}{a^3}\delta_{np}\delta_{nq},$$

$$R3_{p,r} = -\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dmn\pi^2}{ab}(-1)^{(m+n)}\delta_{np}\delta_{mr},$$

$$R4_{p,s} = \sum_{n=1}^{N}\frac{(1-\nu)Dmn\pi^2}{ab}(-1)^{(m+n)}\delta_{np}\delta_{ms} - \sum_{m=1}^{M}(-1)^m\sum_{n=1}^{N}\frac{2(1-\nu)Dn^2\pi^2}{ab}\delta_{np}\delta_{ms},$$

$$R5_{q,q} = \sum_{n=1}^{N}\frac{(1-\nu)Dn\ \pi^2}{a}\delta_{np} + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dm^2\pi^2}{a^3}\delta_{np} + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{2(1-\nu)Dn^2\pi^2}{ab}\delta_{nq},$$

$$R6_{q,r} = -\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dmn\pi^2}{ab}(-1)^m\delta_{nq}\delta_{mr},$$

$$R7_{q,s} = \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dmn\pi^2}{ab}\delta_{nq}\delta_{ms},$$

$$R8_{r,r} = \sum_{m=1}^{M}\frac{(1-\nu)Dm^2\pi^2}{ab}\delta_{mr} + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dn^2\pi^2}{b^3}\delta_{mr} + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{2(1-\nu)Dm^2\pi^2}{ab}\delta_{mr},$$

$$R9_{r,s} = -\sum_{m=1}^{M}\frac{(1-\nu)Dm^2\pi^2}{ab}(-1)^n\delta_{mr}\delta_{ms} - \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dn^2\pi^2}{b^3}(-1)^n\delta_{mr}\delta_{mr}$$
$$-\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{2(1-\nu)Dm^2\pi^2}{ab}(-1)^n\delta_{mr}\delta_{ms},$$

$$R10_{s,s} = \sum_{m=1}^{M}\frac{(1-\nu)Dm^2\pi^2}{ab}\delta_{ms} + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{Dn^2\pi^2}{b^3}\delta_{ms} + \sum_{m=1}^{M}\sum_{n=1}^{N}\frac{2(1-\nu)Dm^2\pi^2}{ab}\delta_{ms},$$

where δ_{ij}=Kronecker delta and is defined as follows:

$$\delta_{ij} = \begin{cases} 1 & \text{if}(i=j) \\ 0 & \text{if}(i\neq j) \end{cases}$$